THE EVALUATION OF BLACKJACK GAMES USING A COMBINED EXPECTATION AND RISK MEASURE

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Abstract

Given a choice of various blackjack games with different expectations and variances, which game should a player select? This paper proposes combining expectation and variance into a single index which reflects the profitability of a given game in terms of return on bankroll. The index is used to compare various types of games and playing styles.

Introduction

By the early 1980's, most skilled blackjack players had come to appreciate the importance of game selection to their profits. Rather than invest the great deal of time and effort required to learn the latest multi-level, multi-parameter count with a complete set of decision indices, players took to seeking out games with better penetration, fewer decks, or which allowed larger betting spreads while playing simple level-one counts.[1] With this emphasis on game selection, more and more players (due primarily to the efforts of Wong[2] and Snyder[3]) learned how to calculate the per hand expectation and variance of the blackjack games in which they had the opportunity to play and hopefully make money. Typically when making a selection, players have favored the game with the greatest percentage expectation per hand, while variance has been used as a measure of what sort of fluctuations from normal players might expect. The emphasis on percentage expectation per hand, however, is not appropriate for players whose wagers (in light of their risk preferences) are limited by their bankroll. For these players, their investment in each hand of blackjack is not limited to the money they have placed on

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[1] Even Ken Uston, originally a strong promoter of multi-level counts with an ace side count, eventually came to this view. "I erroneously suggested to readers in my *Million Dollar Blackjack* that [the Uston Advanced Plus/Minus] was inferior to the Uston APC... Because of the complexity of the Uston APC...I now believe the Advanced Plus/Minus to be a far more practical count," (Uston, 1986), p.93. It should be noted, however, that Uston always emphasized the importance of game selection. See (Uston, 1981).

[2] See, for example, (Wong, 1983), pp.57-60.

[3] See (Snyder, 1987).

the table. In effect, they have invested their entire bankroll in this endeavor, and it is the return on their bankroll that determines which game is most profitable for them.

The task at hand is to develop an index which reflects the profitability of a game in terms of return on bankroll. The analysis will be simplified by the following assumptions:

Assumption 1 — The risk preferences of the player are completely reflected by a maximum allowable probability of ruin.

Assumption 2 — The player uses a fixed betting scheme. That is, prior to entering the game he decides how many units he shall bet at each count. This strategy as well as the dollar value of one unit may not change as the game progresses, even as the player's bankroll fluctuates.

Assumption 3 — The player is playing *at the limit* of his bankroll in whatever game he chooses to play. By this it is meant that the player is unwilling to increase the amount wagered on any bet as he would then consider the game too risky.

Assumptions 1 and 2 taken together clearly rule out betting according to the Kelly criterion as well as other similar strategies, but it is the author's opinion that these assumptions closely reflect the behavior of most skilled players. Although theoretically optimal in many respects, the Kelly criterion is difficult to implement in practice. It subjects the player's bankroll to large fluctuations, requires the player to refrain from betting on negative and neutral counts, and in general makes the skilled player look just like what he is to the casino - a card counter. For these and other reasons, most players (and texts) approach the problem from a probability of ruin standpoint. It is also noted that Assumption 3 may require the player to wager odd amounts such as $102.73, which - while not realistic - greatly simplifies the analysis of the problem and should not have a significant impact on the conclusions which follow.

The balance of the paper shall be divided into two main sections. The first part deals with the mathematical development of expressions for probability of ruin and return on bankroll. An index which reflects the profitability of a blackjack game is suggested at the conclusion of the section. The second part of the paper investigates various applications and presents some general results.

Mathematical Development

As a necessary preliminary to deriving an expression for return on bankroll, a method for finding probability of ruin as a function of expectation, variance, and bankroll must be developed. Unfortunately, the most commonly used expression only applies to games where the possible outcomes of each subgame are restricted to ±1. In this case, the formula for probability of ruin[4] is:

$$r = \frac{1 - S^{b}}{1 - S^{a+b}} \qquad [1]$$

where r = probability of ruin
S = ratio of the probability of winning to the probability of losing a hand
a = the number of units in the initial bankroll
b = the number of units to be won

To generalize this expression, a simple model of a game of blackjack proposed by Griffin[5] will be utilized. If the game is such that the player has an expectation of E dollars per hand with a variance of σ^2, the players experience may be approximated by a coin toss game where the player wins on heads, loses on tails, wagers σ on each hand, and the probability of heads is given by

$$P_{Heads} = \frac{1}{2} + \frac{E}{2\sigma} \qquad [2a]$$

The probability of tails is naturally given by

$$P_{Tails} = \frac{1}{2} - \frac{E}{2\sigma} \qquad [2b]$$

This game has the same expectation and variance as the blackjack game, but it has the important difference that the outcome is always plus or minus σ. Thus, expression [1] may be used to calculate the probability of ruin with σ considered to be the unit bet. Substituting the following expressions into equation [1]

$$S = \frac{P_{Heads}}{P_{Tails}} = \frac{1 + \frac{E}{\sigma}}{1 - \frac{E}{\sigma}} \qquad [3]$$

$$a = \frac{B}{\sigma} \qquad b = \frac{G}{\sigma}$$

where B is the players bankroll (in dollars) and G is the players goal or desired win (in dollars), an equation is obtained for the probability of ruin as a function of E, σ, B, and G:

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[4]This is a well known expression. See, for example, (Uspensky, 1937), pp. 139-140; (Wilson, 1965), pp. 287-288; and (Chambliss & Roginski, 1981), p. 164.

[5]See (Griffin, 1988), pp. 141-142.

$$r = \frac{1 - \left(\dfrac{1 + \frac{E}{\sigma}}{1 - \frac{E}{\sigma}} \right)^{\frac{G}{\sigma}}}{1 - \left(\dfrac{1 + \frac{E}{\sigma}}{1 - \frac{E}{\sigma}} \right)^{\frac{B+G}{\sigma}}} \qquad [4]$$

As a practical matter, this result may be greatly simplified by applying two assumptions. First, if G>>B, equation [4] may be rewritten:

$$r \approx \left(\frac{1 - \frac{E}{\sigma}}{1 + \frac{E}{\sigma}} \right)^{\frac{B}{\sigma}} \qquad [5]$$

This yields quite accurate results for G≥2B. For G=B, a common goal for players, the results are slightly conservative. Given that due to practical factors a player's expectation is probably less than indicated by his paper calculations, conservative results should not be considered undesirable.

Before proceeding, it is convenient to take the natural logarithm of both sides of equation [5]:

$$\ln r \approx \frac{B}{\sigma} \left(\ln \left(1 - \frac{E}{\sigma} \right) - \ln \left(1 + \frac{E}{\sigma} \right) \right) \qquad [6]$$

A second assumption is now applied to [6]. In any blackjack game of interest, it will certainly be the case that $E \ll \sigma$. Using the approximation $\ln(1 + \varepsilon) \approx \varepsilon$, equation [6] becomes

$$\ln r \approx \frac{-2EB}{\sigma^2} \qquad [7]$$

Equation [7] may be, used to give a very simple expression for the probability of ruin:

$$r \approx \mathrm{EXP} \left(\frac{-2EB}{\sigma^2} \right) \qquad [8]$$

As a practical matter, equation [8] should probably be used by most players to calculate probability of ruin.

Often, it is of more interest to be able to calculate a player's required bankroll from his risk preferences and the parameters of the game, in which case [8] may be rewritten:

$$B \approx -\frac{\sigma^2}{2E} \ln r \qquad [9]$$

Many players choose to play at the 5% chance of ruin level, in which case the bankroll requirement is given by:

$$B \approx \frac{3\sigma^2}{2E} \qquad [9a]$$

Returning to the main focus of this discussion, equation [9] may be used to obtain an expression for return on bankroll:

$$\frac{E}{B} \approx -2\left(\frac{E}{\sigma}\right)^2 \frac{1}{\ln r} \qquad [10]$$

So, for some fixed risk preference,

$$\frac{E}{B} \propto \left(\frac{E}{\sigma}\right)^2 \qquad [10a]$$

Equation [10a] suggests using E/σ or perhaps $(E/\sigma)^2$ as a measure of the quality of a blackjack game. At the expense of a few calculator keystrokes, however, an index can be defined which has some intuitive meaning:

$$\text{Index} \equiv \frac{2}{3}\left(\frac{E}{\sigma}\right)^2 \times 10^4 \qquad [11]$$

This index may be interpreted as the expected per hand dollar win for a player with a $10,000 bankroll who will tolerate a 5% chance of ruin. It is assumed that this hypothetical player uses the same betting scheme as was used to calculate E and σ (although he will in general have a different size unit bet). This index will be used to evaluate various games and playing styles in what follows. It will henceforth be referred to as the Total Return Index, or more simply as TR.

Applications and Results

First consider a player whose betting range is constrained by some given minimum and maximum number of units. (However, there is no constraint on the size of the unit.) The players problem is to determine how to size his bets when the constraints are not binding. The result obtained below confirms again the robustness of the idea that one should wager in proportion to one's advantage.

Definition: An *optimal betting strategy* is a betting scheme which maximizes TR subject to the constraints.

Result 1: The player's optimal (fixed) betting strategy has the property that all bets which do not face a binding constraint are sized in proportion to the player's advantage on the upcoming hand.

Proof: It is supposed that the player (through the use of a count) is able to identify N states, each corresponding to a different advantage A_i. The probability of the ith state is denoted P_i. Then

$$E = \sum_i P_i A_i B_i \qquad \sigma^2 = \alpha \sum_i P_i B_i^2 \qquad [12]$$

where B_i is the number of units wagered in state i, and $\alpha \approx 1.21$ is the variance of outcomes of a single hand of blackjack. Maximizing TR is equivalent to maximizing $(E/\sigma)^2$, so an optimal betting scheme must satisfy:

$$\left. \frac{\partial}{\partial B_i} \left(\frac{E}{\sigma} \right)^2 \right|_{B_j = B_{jo}, \forall j} = 0, \quad \forall i \in \Omega \qquad [13]$$

where Ω = { i : constraints are not binding on B_i }
B_{jo} = optimal bet associated with state j

Substituting [12] into [13] and rearranging terms:

$$B_{io} = \beta A_i \ , \qquad \forall i \in \Omega \qquad [14]$$

where $E_o = \sum_i P_i A_i B_{io}$, $\qquad \sigma_o{}^2 = \alpha \sum_i P_i B_{io}{}^2$

and $\beta = \left(\frac{\sigma_o{}^2}{\alpha E_o} \right)$

The expression for the proportionality constant β has some practical application. Suppose the player knows E and σ for a betting strategy which, although probably not optimal, is at least reasonable. The player can then estimate β using E and σ.

By way of example, consider a player who uses a standard level one count, playing in a Reno one deck game with 75% penetration, spreading one-to-two. Until this point, he has been betting two units whenever his advantage was non-negative (i.e. whenever his true count TC≥1). He knows

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$$E = 0.0114 \qquad \sigma = 1.72$$

From this he estimates β:

$$\beta \approx \frac{(1.72)^2}{(1.21)(.0114)} = 214$$

Thus, the player's bet should be a little more than 200 times his advantage or equivalently a little more than twice his percent advantage. This implies he should raise his bet to two units when he has a 1% advantage (TC = +3). Table 1 below shows the complete picture. It gives the player's percent expected return (%E) and the Total Return Index (TR) when the betting scheme requires the wager to be raised to two units at various minimum true counts.

TABLE 1

True Count	1	2	3	4	5
Advantage	0.0	0.5	1.0	1.5	2.0
%E	0.772	0.825	0.844	0.837	0.813
TR	0.295	0.334	0.349	0.344	0.327

%E and TR when bet is raised to 2 units at different minimum true counts.

An interesting feature of this result (the player should raise his bet when the true count rises to +3) is that it does not depend on the level of risk that the player is willing to accept; his risk preferences only determine the size of his unit bet relative to his bankroll.

Although the above table indicates maximum %E point coinciding with the maximum TR point, this is not in general the case, especially when comparing games with different numbers of decks. This point is emphasized in the second result:

Result 2: Games with the same %E will not in general have the same TR. Further, it is most often the case that the game with fewer cards left undealt will be preferred (assuming that the player is not table-hopping).

Argument: This result is phrased weakly, as it will be seen that it is possible to construct exceptions. However, for most cases of practical interest, the result holds. The information contained in Table 2 is typical.

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[6]Values for E and σ, as well as the information required to compute Table 1, were obtained from (Snyder, 1987a), p.41.

Argument: The result is again best demonstrated through a table of results. To simulate table-hopping results, consider a player who leaves the game whenever his advantage $A_i<0$. An approximate frequency table can be constructed from a standard frequency table by considering only those entries such that $A_i \geq 0$ and renormalizing the probabilities over these entries. Table 3 summarizes the results:

TABLE 3

Decks	Penetration	Spread	TR	Decks Undealt
4	85%	1-2	0.78	.60
4	75%	1-3	0.72	1.00
4	65%	1-10	0.49	1.40
6	85%	1-5	0.79	.90
6	75%	1-8	0.60	1.50
6	65%	1-22	0.41	2.10
8	85%	1-12	0.65	1.20
8	75%	1-27	0.43	2.00

Some caution, however, is warranted in drawing comparisons between single and multi-deck games based on these results. The TR index gives of measure of profitability on a per hand basis. There may be a substantial difference in the number of hands played per hour between the table-hopper and stationary player. (It is also noted that the speed of all games is directly affected by the number of players at the table.) In spite of this caveat, table-hopping multi-deck games with good penetration appears to be a good alternative to the single deck game.

Conclusion

By combining expectation and variance into a single measure, this paper developed an index which provides the player with a more appropriate indicator of the profitability of a blackjack game. The TR index achieves this by not only considering the percentage return on an average bet, but also by considering how the size of an average bet is affected by the variance of the game. This index was then used to obtain several results which are hoped to be general enough to be of use to the average player.

Other possible applications of the TR index may be considered. For example, can card counting systems be improved by eliminating some high variance plays which are only marginally profitable?[8] Another idea would be an analysis of cover plays: Is it possible that some cover plays, such as always insuring a blackjack (which costs expectation, but lowers variance), really don't impact a players return on bankroll in any significant way? What are the true costs associated with high variance cover strategies such as depth charging, opposition betting, and negative progressions?[9] It seems that even in the narrow area considered in this paper, the game of blackjack still possesses many unanswered questions.

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[8] Of interest in this regard is (Friedman, 1980), which introduces the idea of deriving playing strategies using a logarithmic utility function.

[9] For discussion of these and other strategies, see (Snyder, 1983) and (Malmuth, 1987).